# 02465: Introduction to reinforcement learning and control

Exploration and Bandits

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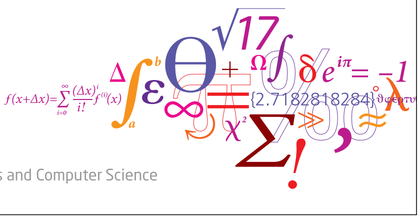


DTU Compute
Department of Applied Mathematics and Computer Science

## Lecture Schedule

Dynamical programming

1. The finite-horizon decision problem
   2 February
2. Dynamical Programming
   9 February
3. DP reformulations and introduction to Control
   16 February

Control

4. Discretization and PID control
   23 February
5. Direct methods and control by optimization
   1 March
6. Linear-quadratic problems in control
   8 March
7. Linearization and iterative LQR
   15 March

Reinforcement learning

8. **Exploration and Bandits**
   22 March
9. Policy and value iteration
   5 April
10. Monte-carlo methods and TD learning
    12 April
11. Model-Free Control with tabular and linear methods
    19 April
12. Eligibility traces and value-function approximations
    26 April
13. Q-learning and deep-Q learning
    3 May

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

**Reading material:**

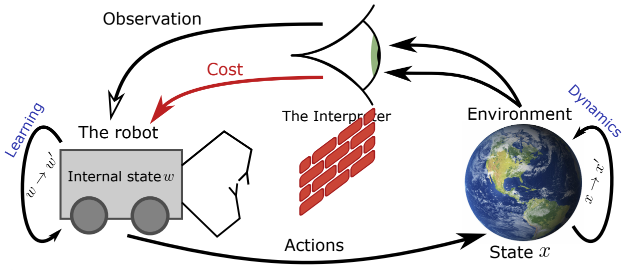
- [SB18, Chapter 1; Chapter 2-2.7; 2.9-2.10] Only as background

**Learning Objectives**

- Exploration/exploitation problem
- Bandits as a simplified reinforcement learning setting
- Formalizing the bandit problem
- Algorithms for solving the bandit problem

## Housekeeping

- Project evaluations are online
- Many singleton groups – try to merge groups (use Discord or the classrooms)
  - No advantage to working alone.
- Sorry for messing up the quiz last time; **Option a** was correct (I remembered it as false and didn't read it during class..); I have uploaded an improved version of the quiz in the slides and will refrain from double-negation.

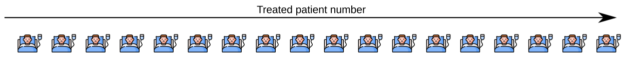


- Dynamics of world not known
- Simultaneously **learn the environment** and **maximize expected reward**
- Balance **exploration** and **exploitation**

**Bandit studies this in an idealized setting**

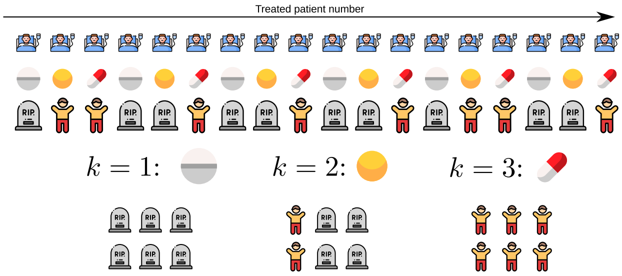
## Bandits, examples

- Suppose you have a large number of patients $t = 1, 2, \dots$ with the same disease
- You have access to $k$ drugs $a = 0, 1, \dots, k-1$ with different outcome probabilities
- Outcome of treatment is either that the patient recovers, $R_t = 1$, or not $R_t = 0$
- Goal is to maximize $\sum_{t=1}^{T} R_t$



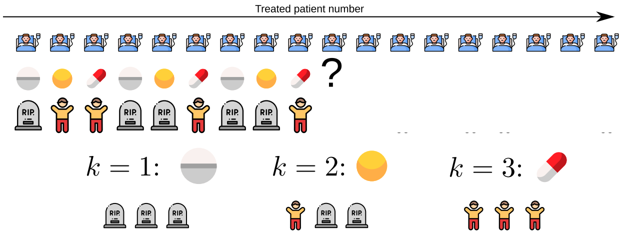
## Idea 1: Statistics!

- Divide first $T$ patients into $K$ groups of $S = \frac{T}{K}$ patients
- Administer drugs to each group
- observe results



## Bandit approach

- After $t-1$ choices of actions $A_1, \dots, A_{t-1}$ and observed rewards $R_1, \dots, R_{t-1}$
- Decide next action $A_t$ to maximize reward
- **Bandit assumption:** Action $A_t$ only affects $R_t$
  - Personalized medicine
  - Evaluating similar, approved drugs (low risk)
  - SMART trials/JITAIs



## Example: An opinion columnist

Suppose you are writing for a major newspaper which relies on social media to get as many reads as possible. You can choose between 5 headlines, and your job is to get as many clicks as possible:

- $k = 0$: "With less destructive nukes on the way, it's time for the left to say good-bye to those annoying non-proliferation treaties. "
- $k = 1$: "Opinion | The upside of nuclear war? Making popcorn without a microwave."
- $k = 2$: "Joe Biden has prevented a nuclear holocaust. But how will that play with suburban moms this fall?"
- $k = 3$: "Opinion | Nuclear war may not be woke. But it's not a war crime."
- $k = 4$: "Opinion | With rising temperatures, would a nuclear winter really be that bad?"

But which one to choose?

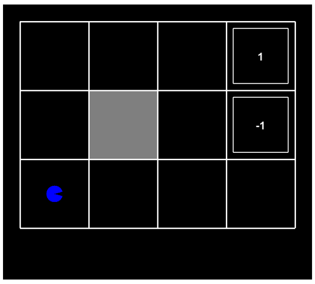
## Example: An opinion columnist

- For each exposure $t = 1, 2, \dots$ on twitter, selects a headline $A_t = 0, \dots, k-1$
- Observe whether the user clicks the story $R_t \in \{0, 1\}$
- Use this to select the next headline for the next user $A_{t+1} = a$
- **You want to maximize total clicks, knowing the story has a finite lifespan:**

$$\sum_{t=1}^{\text{2-3 days?}} R_t$$

## Looking ahead: Reinforcement learning

- In a state $s$, select optimal action $a$, then observe what reward we get
- It is like a bandit problem in each state (but more about that in a few weeks)



## Many types of bandits

Sequentially take decisions $A_1, A_2, \dots$ and observe rewards $R_1, R_2, \dots$

Stationary In a stationary bandit the reward distribution does not change

Nonstationary The environment can change (but not as consequence of our actions)

Contextual You get a bit of information to make your decision

Structured Reward of different arms can be inferred from each other (Bayesian black box optimization)

## Stationary bandits

- Action at time step $t = 1, 2, \dots$ is $A_t$
- Reward is $R_t$
- Observations available to make action at $t$:

$$H_t = (A_1, R_1, A_2, R_2, \dots, A_{t-1}, R_{t-1})$$

- Actions are generated from a **policy** $\pi$ which we learn based on $H_t$:

$$A_t \sim \pi_t(\cdot)$$

- **Value** of an action is

$$q_*(a) = \mathbb{E}\left[R_t | A_t = a\right], \quad a = 0, \dots, K-1$$

- Optimal strategy at $t$ is to select action with highest value
- Our learned estimate of $q_*(a)$ at time $t$ is $Q_t(a)$

  Exploit Select action $a$ with **highest** estimate of $Q_t(a)$
  Explore Do something else to **learn** more about $Q_t(a)$

- Note bandit methods can be classified according to what they learn about $Q_t(a)$

## Bandit objective and definitions

Objective 1: Average reward at time $t$ and total reward up to time $T$

$$\mathbb{E}_\pi\left[q_*(a_t)\right], \quad \sum_{t=1}^{T} \mathbb{E}_\pi\left[q_*(a_t)\right]$$

Optimal value and optimal action

$$V^* = \max_a \left[q_*(a)\right], \quad a_t^* = \arg\max_a \left[q_*(a)\right]$$

Objective 2: Fraction optimal actions

$$P_\pi(A_t = a_t^*)$$

Gab

$$\Delta_a = V^* - q_*(a)$$

Objective 3: Cumulative regret

$$l_t = \mathbb{E}\left[V^* - q_*(a_t)\right], \quad L_T = \sum_{t=1}^{T} l_t$$

**Goal is to maximize cumulative reward ↔ minimize total regret**

## Quiz: What is the regret?

- Reward $R_t = 1$ on win and $R_t = 0$ on loss.
- The win probabilities are shown by horizontal lines
- What is the regret for a policy which always select $a = 3$? ($\pi(a = 3) = 1$)

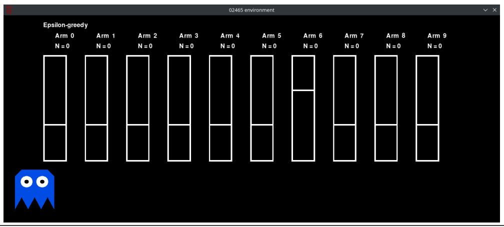
$$l_t = \mathbb{E}\left[V^* - q_*(a_t)\right], \quad V^* = \max_a \left[q_*(a)\right]$$

**a.** It is a random quantity (either zero or 1)

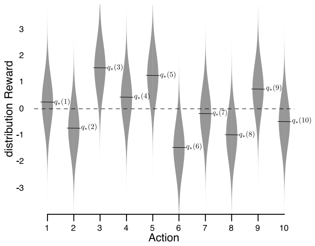
**b.** It depends on how many actions we have taken

**c.** It is about $\frac{1}{3}$

**d.** It is about $-\frac{2}{3}$



## The $k = 10$-armed testbed



- Let $k = 10$ and select each $q_*(a) \sim N(\mu = 0, \sigma^2 = 1)$
- for each action $a$, select reward

$$R_t | a \sim \mathcal{N}(\mu = q_*(a), \sigma^2 = 1)$$

- Let each agent interact for a number of **steps** $\sim 1000$
- Repeat procedure for $2000$ runs to calculate average agent performance

## Making it practical: A bandit problem

```python
# bandits.py
class BanditEnvironment(Env):
    def __init__(self, k : int):
        super().__init__()
        self.observation_space = Discrete(1)  # Dummy observation space with a single o
        self.action_space = Discrete(k)       # The arms labelled 0,1,...,k-1.
        self.k = k  # Number of arms


    def reset(self):
        raise NotImplementedError("Implement the reset method")

    def bandit_step(self, a):
        reward = 0 # Compute the reward associated with arm a
        regret = 0 # Compute the regret, by comparing to the optimal arms reward.
        return reward, regret

    def step(self, action):
        reward, average_regret = self.bandit_step(action)
        info = {'average_regret': average_regret}
        return None, reward, False, False, info
```

## Action-value method

**Idea: approximate $q_*(a)$ by keeping track of $Q_t(a)$**

$$Q_t(a) \doteq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbf{1}_{A_i = a}}{\sum_{i=1}^{t-1} \mathbf{1}_{A_i = a}} = \frac{S_t(a)}{N_t(a)}$$

Explore with probability $\epsilon$

- Action selection $\pi$
  - With probability $\epsilon$ select random action
  - With probabilty $1 - \epsilon$ select $a^* = \arg\max_a Q_t(a)$
- As only one entry $A_t$ of $Q_t$ change at a time track number of times $a$ was selected $n = N(a)$:

$$Q_n \doteq \frac{R_1 + R_2 + \cdots + R_{n-1}}{n-1} = \frac{S_n(a)}{N(a)} \tag{1}$$

One can show that:

$$Q_{n+1} = Q_n + \frac{1}{n}\left[R_n - Q_n\right]$$

  - Given observed $a = A_t$, $r = R_t$ update:

## Simple action-value bandit algorithm

**A simple bandit algorithm**

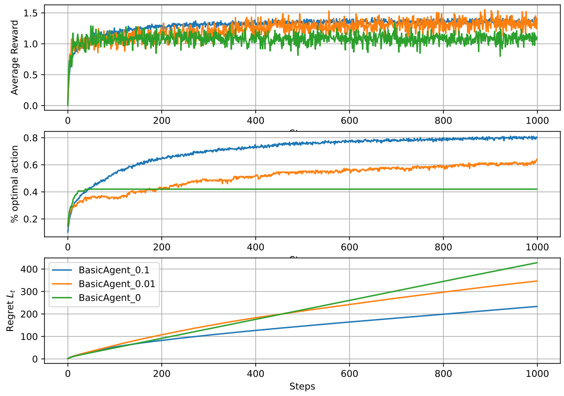
```
Initialize, for $a = 1$ to $k$:
    $Q(a) \leftarrow 0$
    $N(a) \leftarrow 0$

Loop forever:
    $A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \quad \text{(breaking ties randomly)} \\ \text{a random action} & \text{with probability } \varepsilon \end{cases}$
    $R \leftarrow bandit(A)$
    $N(A) \leftarrow N(A) + 1$
    $Q(A) \leftarrow Q(A) + \frac{1}{N(A)}\big[R - Q(A)\big]$
```

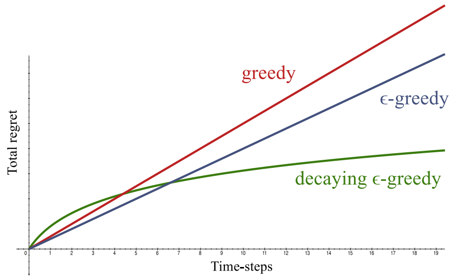
$$Q_{n+1} = Q_n + \frac{1}{n}\Big[R_n - Q_n\Big]$$

## Results: action-value bandit

Evaluated on StationaryBandit_0 for 150 episodes



## Regret asymptotics



- Fixed-$\varepsilon$ algorithms have linear regret
- With decreasing $\varepsilon$ it is possible to get sub-linear regret, **but only by assuming we know things about the reward distribution**
    - Theoretically best possible bandit method has logarithmic regret.

## Confidence-bound methods

- Estimate an upper confidence bound $\hat{U}_t(a)$ for $q_*(a)$ st.

$$q_*(a) \leq \hat{U}_t(a) + Q_t(a)$$

  with high probability
- Generally
    - If $N_t(a)$ low → $\hat{U}_t(a)$ high
    - If $N_t(a)$ high → $\hat{U}_t(a)$ low
- Select actions to minimize

$$\arg\min_a \Big[\hat{U}_t(a) + Q_t(a)\Big]$$

## UCB1

$$A_t = \operatorname*{argmax}_a \left[ Q_t(a) + c\sqrt{\frac{\ln t}{N_t(a)}} \right]$$

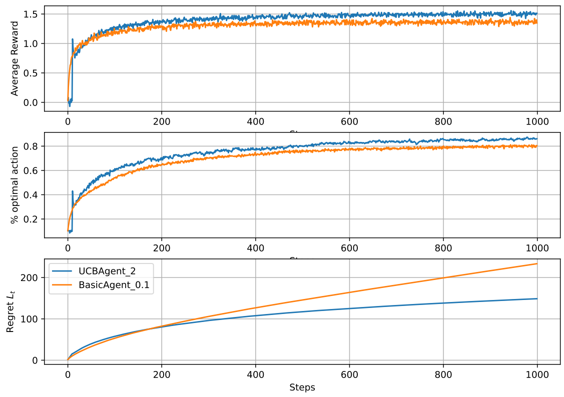
Asymptotic logarithmic regret when $R_t \in [0, 1]$

$$\lim_{t\to\infty} L_t \leq \sum_{a \neq a^*, \Delta_a > 0} \left( \frac{4 \ln t}{\Delta_a} + 2\Delta_a \right)$$

- The variant UCB-normal obtains logarithmic regret on normal reward distributions

## Results

Evaluated on StationaryBandit_0 for 2000 episodes



## Quiz: How does UCB explore?

Consider the update rule for UCB1:

$$A_t = \operatorname*{argmax}_a \left[ Q_t(a) + c\sqrt{\frac{\ln t}{N_t(a)}} \right]$$

Which one of the following statements is true about UCB1?

**a.** UCB1 requires that the rewards are positive

**b.** If one arm give a much higher reward than the other, UCB1 will eventually only select this arm

**c.** If one arm is much, much worse than the others, UCB1 will eventually stop selecting that arm

**d.** It is possible to predict which arms UCB1 will select $k$ steps in the future

**e.** At least one of the upper-confidence estimates $\hat{U}_t(a)$ will converge to 0.

**f.** Don't know.

## Non-stationary bandits

- These is a (hidden) state $S_t$ which evolves as:

$$P(S_{t+1}, R_t | S_t = s, A_t = a) = P(S_{t+1} | S_t = s) P(R_t | S_t = s, A_t = a)$$

- Example: Add normal noise to $q_*(a)$ at each time step
- One idea is to replace $\frac{1}{n}$ with $\alpha_t(a)$ and use scheduling:

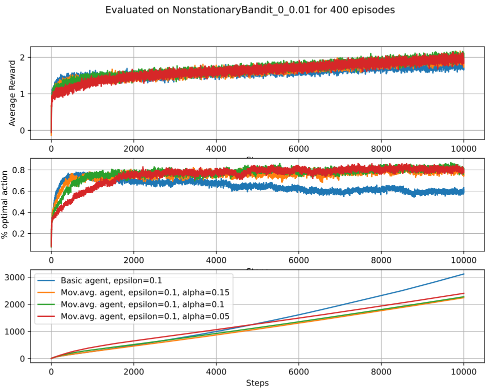
$$\text{Previous update: } Q_{n+1} = Q_n + \frac{1}{n}[R_n - Q_n]$$

$$\text{New update: } Q_{n+1} = Q_n + \alpha [R_n - Q_n]$$

  - Constant $\alpha$ means fast adaption but no convergence
  - Typically chose

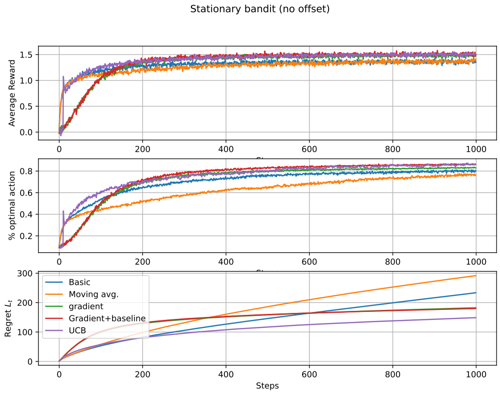
$$\sum_{n=1}^{\infty} \alpha_n(a) = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n^2(a) < \infty$$

## Results



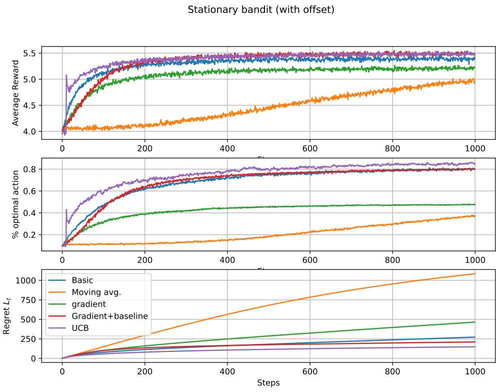
Evaluated on NonstationaryBandit_0_0.01 for 400 episodes

## Results



Stationary bandit (no offset)

## Results



Stationary bandit (with offset)

Richard S. Sutton and Andrew G. Barto.
*Reinforcement Learning: An Introduction.*
The MIT Press, second edition, 2018.
(Freely available online).

## Appendix: Probability-matching methods

- Our goal is to find the optimal probability distribution $\pi$
- We can parameterize any distribution as

$$\pi(a) = \frac{e^{H_a}}{\sum_{b=1}^{k} e^{H_b}}$$

for a weight-vector $H \in \mathbb{R}^k$

- Optimal $\pi$ is the one maximizing expected reward

$$\mathbb{E}_\pi\left[R_t\right] = \sum_a \pi_t(a;H) q_*(a) = E(H)$$

- This is a function of $H$
- Let's just do gradient descent, WCGW?

$$H_{t+1} \leftarrow H_t - \alpha \nabla_H E(H)$$

## Gradient bandit: Derivation

$$\frac{\partial}{\partial H} E(H) = \sum_a \pi(a;H) q^*(a) \frac{\partial \log \pi(a;H)}{\partial H} \tag{2}$$

We can sample from $\pi(a)$ and then our environment will give an estimate of $q^*(a)$

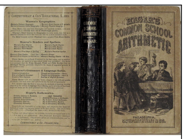
$$\sum_a \pi(a;H) q^*(a) \frac{\partial \log \pi(a;H)}{\partial H} \approx \frac{1}{S}\sum_{s=1}^{S} R_t(a_s) \frac{\partial \log \pi(a_s;H)}{\partial H} \tag{3}$$

- Nobody has told us we cannot use $S = 1$

$$\nabla E(H) \approx R_t \frac{\partial \log \pi(a_t;H)}{\partial H}$$

$$\begin{aligned} H_{t+1}\left(A_t\right) &\doteq H_t\left(A_t\right) + \alpha R_t \left(1 - \pi_t\left(A_t\right)\right), && \text{and} \\ H_{t+1}(a) &\doteq H_t(a) - \alpha R_t \pi_t(a), && \text{for all } a \neq A_t \end{aligned}$$

## Math facts used in derivation



Kullback-Leibner divergence Given discrete probability distribution $p$ and $q$:

$$\mathrm{KL}[p;q] = \sum_{i=1}^{n} p(x_i) \log \frac{q(x_i)}{p(x_i)}$$

The logarithm trick for $q(x,\theta) > 0$

$$\frac{\partial}{\partial \theta} \int q(x,\theta) f(x) dx = \int q(x,\theta) \frac{\partial \log q(x,\theta)}{\partial \theta} f(x) dx$$

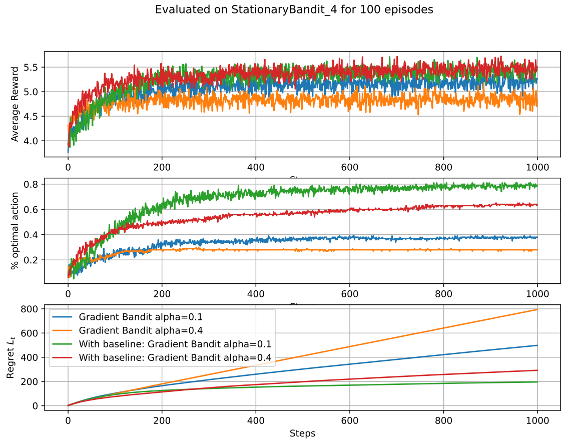
## Gradient bandits

- Let $\bar{R}_t$ be the average reward over $0, \dots, t-1$
- Update weights as

$$\begin{aligned} H_{t+1}\left(A_t\right) &\doteq H_t\left(A_t\right) + \alpha \left(R_t - \bar{R}_t\right) \left(1 - \pi_t\left(A_t\right)\right), && \text{and} \\ H_{t+1}(a) &\doteq H_t(a) - \alpha \left(R_t - \bar{R}_t\right) \pi_t(a), && \text{for all } a \neq A_t \end{aligned}$$

- Why? **legal** because they do not change the gradient, **sensible** because they can reduce variance/promote exploration
- To my knowledge, no theoretical analysis exists
- This gradient-trick is basis of **policy gradient** methods for reinforcement learning

## Results


Evaluated on StationaryBandit_4 for 100 episodes
Average Reward
% optimal action
Regret $L_t$
Steps
Gradient Bandit alpha=0.1
Gradient Bandit alpha=0.4
With baseline: Gradient Bandit alpha=0.1
With baseline: Gradient Bandit alpha=0.4